JOURNAL OF APPROXIMATION THEORY **87**, 179–193 (1996)
ARTICLE NO. 0100

Uniform Convergence of Lagrange Interpolation Based on the Jacobi Nodes

George Kvernadze

Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, New Mexico 87131
E-mail: gkverna@math.unm.edu

Communicated by Paul Nevai

Received April 20, 1995; accepted in revised form November 28, 1995

Necessary and sufficient conditions are obtained for a continuous function guaranteeing the uniform convergence on the whole interval $[-1, 1]$ of its Lagrange interpolant based on the Jacobi nodes. The conditions are in terms of Λ-variation, Φ-variation, the modulus of variation, and the Banach indicatrix of a function. © 1996 Academic Press, Inc.

INTRODUCTION

1. Throughout this paper we use the following notations: N and Z_+ are the sets of positive and nonnegative integers, respectively. M is the space of bounded functions on $[-1, 1]$ and C is the space of continuous functions on $[-1, 1]$ with uniform norm $\|\cdot\|_C$. By $\omega(\delta, f)$ we denote the modulus of continuity of $f \in C$ on $[-1, 1]$, i.e.,

$$\omega(\delta, f) = \max\{|f(x+h) - f(x)| : x, x+h \in [-1, 1], |h| \leqslant \delta\}. \qquad (1)$$

The function ρ is called a Jacobi weight if $\rho(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha > -1$, and $\beta > -1$. If ρ is a Jacobi weight, then by $\sigma(\rho) = (P_n^{(\alpha, \beta)}(x))_{n=0}^\infty$ we denote the corresponding system of orthogonal polynomials $P_n^{(\alpha, \beta)}(x) = \gamma_n(\alpha, \beta)\, x^n + \textit{lower degree terms}$, $\gamma_n(\alpha, \beta) > 0$, i.e.,

$$\int_{-1}^{1} P_n^{(\alpha, \beta)}(x)\, P_m^{(\alpha, \beta)}(x)\, \rho(x)\, dx = \delta_{nm}, \qquad n \neq m.$$

We assume that they are normalized by the condition $P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$, $n \in N$. The system $\sigma(\rho)$ is defined uniquely and is called the Jacobi system of polynomials.

0021-9045/96 $18.00

By $x(k, n) = x_{k,n} = x_{k,n}^{(\alpha,\beta)}$, $k = 1, ..., n$, we denote the zeros of a polynomial $P_n(x) = P_n^{(\alpha,\beta)}(x)$ arranged in decreasing order. It is known [19, Thm. 3.3.1, p. 44] that all zeros of $P_n(x)$ are real and distinct and belong to $(-1, 1)$.

For a given Jacobi weight ρ and a given function $f \in M$, the corresponding Lagrange interpolating polynomial is denoted by $L_n^{(\alpha,\beta)}(f)$. Hence,

$$L_n^{(\alpha,\beta)}(x_{k,n}^{(\alpha,\beta)}, f) = f(x_{k,n}^{(\alpha,\beta)})$$

for $k = 1, 2, ..., n$, and we can write

$$L_n(x, f) = L_n^{(\alpha,\beta)}(x, f) = \sum_{k=1}^{n} f(x_{k,n}^{(\alpha,\beta)})\, l_{k,n}^{(\alpha,\beta)}(x), \tag{2}$$

where the fundamental polynomials $l_{k,n}^{(\alpha,\beta)}$ are defined by

$$l_{k,n}(x) = l_{k,n}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/(P_n'^{(\alpha,\beta)}(x_{k,n}^{(\alpha,\beta)})(x - x_{k,n}^{(\alpha,\beta)})).$$

By $U = U^{(\alpha,\beta)}$ we denote the class of functions, defined on the interval $[-1, 1]$, for which the sequence of Lagrange interpolating polynomials for indices α and β is uniformly convergent on the whole segment $[-1, 1]$, i.e.,

$$\lim_{n\to\infty} \|L_n^{(\alpha,\beta)}(\cdot, f) - f\|_C = 0.$$

By K we denote positive constants, possibly depending on indices α and β, and in general distinct in different formulas. For positive quantities A_n and B_n, probably depending on some other variables as well, we write $A_n = o(B_n)$ and $A_n = O(B_n)$, if $\lim_{n\to\infty} A_n/B_n = 0$ and $\sup_{n\in N} A_n/B_n < \infty$, respectively. For quantities A and B, depending on some variables, we write $A \sim B$ if the ratio A/B is between two positive constants, independent of the variables.

The following notions are some generalizations of the notion of bounded variation of a function.

DEFINITION 1 [4]. Let $f \in C$. Then the Banach indicatrix $N(y, f)$ of f is defined as follows: for every $y \in (-\infty, \infty)$, $N(y, f)$ is equal to the number (finite or infinite) of solutions of equations $f(x) = y$.

Banach [4, Thm. 2, p. 228] proved that a continuous function f has a bounded variation if and only if $N(y, f)$ is integrable on $(m(f), M(f))$, where $m(f) = \min\{f(x) : x \in [-1, 1]\}$ and $M(f) = \max\{f(x) : x \in [-1, 1]\}$.

DEFINITION 2 [27]. Let Φ be a strictly increasing continuous function on $[0, \infty)$ and $\Phi(0)=0$. A function f is said to have Φ-bounded variation on $[-1, 1]$, i.e., $f \in V_\Phi$, if

$$v_\Phi(f) = \sup_{\Pi} \sum_{k=1}^{n} \Phi(|f(x_k) - f(x_{k-1})|) < \infty,$$

where $\Pi = \{-1 \leqslant x_0 < x_1 < \cdots < x_n \leqslant 1\}$ is an arbitrary partition of $[-1, 1]$.

If $\Phi(x) = x$, then V_Φ coincides with the Jordan class V of functions of bounded variation and when $\Phi(x) = x^p$, $p > 1$, it coincides with the Wiener [26] class V_p.

DEFINITION 3 [6]. Let $f \in M$. The modulus of variation of the function f is called the function $v(n, f)$ defined for $n \in Z_+$ as follows: $v(0, f) = 0$, while for $n \geqslant 1$

$$v(n, f) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(x_{2k+1}) - f(x_{2k})|,$$

where Π_n is an arbitrary system of n disjoint subintervals (x_{2k}, x_{2k+1}), $k = 0, 1, ..., n-1$, of the interval $[-1, 1]$.

If $v(n)$, $n \in N$, is a nondecreasing upwards convex function and $v(0) = 0$, then we call $v(n)$ the modulus of variation.

The class of functions which satisfy the relation $v(n, f) = O(v(n))$ will be denoted by $V[v]$.

In particular, $V[1] = V$.

DEFINITION 4 [24]. Let $\Lambda = (\lambda_k)_{k=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$. A function f is said to have Λ-bounded variation on $[-1, 1]$, i.e., $f \in \Lambda BV$, if

$$v_\Lambda(f) = \sup_{\Pi} \sum_{k=1}^{n} \frac{|f(x_{2k+1}) - f(x_{2k})|}{\lambda_k} < \infty,$$

where Π is an arbitrary system of disjoint intervals $(x_{2k}, x_{2k+1}) \subset [-1, 1]$.

If $\lambda_k = 1$, $k \in N$, then obviously $\Lambda BV = V$.

2. It is well known (cf. [21, Section 8.1.2, p. 469]) that continuity of a function on $[-1, 1]$ alone is not sufficient to imply the uniform convergence of its Lagrange interpolant based on the Jacobi nodes (or at any other nodes either).

In the present paper we examine those conditions on the variation of a continuous function which guarantee its Lagrange interpolant's uniform convergence, and whether these conditions are definitive.

Here we represent a brief review of the question. As a corollary from his more general theorem, Berman [5, Thm. 4, p. 12] obtained the following result: Let $-1<\alpha<0$ and $-1<\beta<0$. Then Lagrange interpolant (2) tends to $f(x)$ for every $x\in(-1,1)$ whenever $f\in C\cap V$. For the special case $\alpha=\beta=-1/2$ this result belongs to Krylov [10, Thm. 2, p. 364]. Later, Geronimus [7, Thm. 10, p. 557] in particular proved that the above mentioned result of Berman is valid for $\alpha>-1$ and $\beta>-1$, and for every $x\in(-1,1)$.

Kel'zon [8] showed that the pointwise convergence of Lagrange interpolant (2) to the function f being interpolated is guaranteed for $f\in V_p$, $p\geqslant 1$, if $-1<\alpha<1/p-1/2$ and $-1<\beta<1/p-1/2$. Nevai [13, Thm. 7, p. 126] studied conditions of Lagrange interpolation uniform convergence on $[a,b]\subset(-1,1)$ for a function $f\in C\cap V_\Phi$ for arbitrary $\alpha>-1$ and $\beta>-1$. Pilipchuk [15, Thm. 1, p. 40] continued this investigation of the problem. He obtained a condition in terms of the modulus of variation of a function $f\in C$ guaranteeing its Lagrange interpolant's uniform convergence strictly inside $[-1,1]$.

Regarding uniform convergence conditions on the whole interval $[-1,1]$, the first result was obtained by Vértesi [22, Thm. 3.1, p. 24]: If $-1<\alpha<1/2$ and $-1<\beta<1/2$, then the following inclusion holds:

$$C\cap V\subset U. \tag{3}$$

He also showed [23, Thm. 3.2, p. 421] that if $\max(\alpha,\beta)=1/2$, then conclusion (3) does not hold.

Kel'zon [9, Thm. p. 21] generalized the result mentioned above for V_p classes of functions: If $-1<\alpha<1/p-1/2$ and $-1<\beta<1/p-1/2$, then the following inclusion holds:

$$C\cap V_p\subset U. \tag{4}$$

X. Sun estimated the convergence rate of the Lagrange interpolant of $f\in C\cap\Lambda_t BV$ for $\Lambda_t=(k^t)_{k=1}^\infty$. From this result we get the following corollary [20, Sect. 5.2 and 5.3, p. 83]: Let $-1<\alpha<1/2$ and $-1<\beta<1/2$. In addition, suppose $\Lambda_t=(k^t)_{k=1}^\infty$, where $t\in[0,1]$. Then the inclusion

$$C\cap\Lambda_t BV\subset U$$

holds if $t<1/2-q$ when $q>-1/2$, and if $t=1$ when $q=-1/2$, where q is defined by (5). X. Sun also mentioned that his result includes the result of Kel'zon (4).

In the present paper we examine those conditions on the variation of a continuous function which guarantee its Lagrange interpolant's uniform convergence, and whether these conditions are definitive.

Here we represent a brief review of the question. As a corollary from his more general theorem, Berman [5, Thm. 4, p. 12] obtained the following result: Let $-1<\alpha<0$ and $-1<\beta<0$. Then Lagrange interpolant (2) tends to $f(x)$ for every $x\in(-1,1)$ whenever $f\in C\cap V$. For the special case $\alpha=\beta=-1/2$ this result belongs to Krylov [10, Thm. 2, p. 364]. Later, Geronimus [7, Thm. 10, p. 557] in particular proved that the above mentioned result of Berman is valid for $\alpha>-1$ and $\beta>-1$, and for every $x\in(-1,1)$.

Kel'zon [8] showed that the pointwise convergence of Lagrange interpolant (2) to the function f being interpolated is guaranteed for $f\in V_p$, $p\geqslant 1$, if $-1<\alpha<1/p-1/2$ and $-1<\beta<1/p-1/2$. Nevai [13, Thm. 7, p. 126] studied conditions of Lagrange interpolation uniform convergence on $[a,b]\subset(-1,1)$ for a function $f\in C\cap V_\Phi$ for arbitrary $\alpha>-1$ and $\beta>-1$. Pilipchuk [15, Thm. 1, p. 40] continued this investigation of the problem. He obtained a condition in terms of the modulus of variation of a function $f\in C$ guaranteeing its Lagrange interpolant's uniform convergence strictly inside $[-1,1]$.

Regarding uniform convergence conditions on the whole interval $[-1,1]$, the first result was obtained by Vértesi [22, Thm. 3.1, p. 24]: If $-1<\alpha<1/2$ and $-1<\beta<1/2$, then the following inclusion holds:

$$C\cap V\subset U. \tag{3}$$

He also showed [23, Thm. 3.2, p. 421] that if $\max(\alpha,\beta)=1/2$, then conclusion (3) does not hold.

Kel'zon [9, Thm. p. 21] generalized the result mentioned above for V_p classes of functions: If $-1<\alpha<1/p-1/2$ and $-1<\beta<1/p-1/2$, then the following inclusion holds:

$$C\cap V_p\subset U. \tag{4}$$

X. Sun estimated the convergence rate of the Lagrange interpolant of $f\in C\cap \Lambda_t BV$ for $\Lambda_t=(k^t)_{k=1}^\infty$. From this result we get the following corollary [20, Sect. 5.2 and 5.3, p. 83]: Let $-1<\alpha<1/2$ and $-1<\beta<1/2$. In addition, suppose $\Lambda_t=(k^t)_{k=1}^\infty$, where $t\in[0,1]$. Then the inclusion

$$C\cap \Lambda_t BV\subset U$$

holds if $t<1/2-q$ when $q>-1/2$, and if $t=1$ when $q=-1/2$, where q is defined by (5). X. Sun also mentioned that his result includes the result of Kel'zon (4).

THEOREM 4. *Let $f \in C$. If the condition*

$$\int_{m(f)}^{M(f)} \int_0^{N(f,\, y)-1} (t+1)^{q-1/2}\, dt\, dy < \infty \tag{10}$$

holds, then $f \in U$.

Condition (10) is definitive in the following sense.

THEOREM 5. *Let $N(y)$ be a positive strictly decreasing continuous function on* $(0, 1)$. *If*

$$\int_0^1 \int_0^{N(y)-1} (t+1)^{q-1/2}\, dt\, dy = \infty, \tag{11}$$

then there exists $f_0 \in C$ such that $N(y, f_0) \leqslant 2N(y)$ and $f_0 \notin U$.

PRELIMINARIES AND LEMMAS

Below we present some basic formulas and lemmas which are necessary in what follows:

$$P_n^{(\alpha,\, \beta)}(x) = (-1)^n\, P_n^{(\beta,\, \alpha)}(-x), \tag{12}$$

$$P_n^{(\alpha,\, \beta)}(1) = \binom{n+\alpha}{n} \sim n^{\alpha}, \tag{13}$$

$$|P_n'^{(\alpha,\, \beta)}(x_{k,\, n})| \sim k^{-\alpha-3/2} n^{\alpha+2} \qquad (x_{k,\, n} \in [0, 1]), \tag{14}$$

$$\operatorname{arc\,cos} x_{k,\, n} \sim n^{-1}(k\pi + O(1)) \qquad (k = 1, ..., n); \tag{15}$$

here $\alpha > -1$, $\beta > -1$, and $n \in N$. (Regarding (12), (14), and (15) see [19, formula (4.1.3), p. 59 and Thm. 8.9.1, p. 238]).

LEMMA 1 [9, p. 22]. *Let $\alpha > -1$, $\beta > -1$, $x \in [-1, 1]$, and $n \in N$. If m is an integer such that $x_{m+1,\, n} < x \leqslant x_{m,\, n}$ (if $x > x_{1,\, n}$ or $x < x_{n,\, n}$, set $m = 0$ or $m = n$, respectively), then*

$$\begin{aligned} \left| \sum_{i=1}^{k} l_{i,\, n}(x) \right| &\leqslant |l_{k,\, n}(x)| \qquad (k = 1, ..., m), \\ \left| \sum_{i=k}^{n} l_{i,\, n}(x) \right| &\leqslant |l_{k,\, n}(x)| \qquad (k = m+1, ..., n). \end{aligned} \tag{16}$$

LEMMA 2 [20, Lemma 3.1, p. 77; Lemma 3.2, p. 78]. *Let* $x_{m_n, n}$ *be the zero of* $P_n^{(\alpha, \beta)}(x)$ *which is closest to* $x \in [-1, 1]$. *Then we have*

$$l_{k, n}(x) = O(|k-m|^{q-1/2}) \qquad (k \neq m), \tag{17}$$

$$l_{m, n}(x) = O(1), \tag{18}$$

$$x - x_{k, n} = O\left(\frac{|k-m|\,(1-x^2)^{1/2}}{n} + \frac{|k-m|^2}{n^2}\right) \qquad (k \neq m), \tag{19}$$

$$x - x_{m, n} = O(1/n), \tag{20}$$

uniformly for $x \in [-1, 1]$, *where* $k = 1, 2, ..., n$, $n \in N$, $m = m_n$, *and* q *is defined by* (5).

LEMMA 3 [16, Thm. 3, p. 114]. *Let* ΛBV *and* ΓBV *be Waterman's classes defined by sequences* $\Lambda = (\lambda_k)_{k=1}^{\infty}$ *and* $\Gamma = (\gamma_k)_{k=1}^{\infty}$, *respectively. Then the inclusion* $\Lambda BV \subset \Gamma BV$ *holds if and only if*

$$\sup_{n \in N} \left(\sum_{k=1}^{n} 1/\gamma_k \right) \left(\sum_{k=1}^{n} 1/\lambda_k \right)^{-1} < \infty.$$

DEFINITION 5 [25]. Let $\Lambda_e = (\lambda_{k+e})_{k=1}^{\infty}$, $e \in N$, where the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$ satisfies the conditions of Definition 4. A function $f \in \Lambda BV$ is said to be continuous in Λ-variation, i.e., $f \in \Lambda_C BV$, if $v_{\Lambda_e}(f) = o(1)$.

LEMMA 4 [17, Thm. 1, p. 88]. *Let a sequence* $\Lambda = (\lambda_k)_{k=1}^{\infty}$, *satisfying the conditions of Definition* 4, *be such that* $\lim_{k \to \infty} \lambda_k / \lambda_{2k}$ *exists. Then* $\Lambda BV = \Lambda_C BV$ *if and only if this limit is less then one.*

LEMMA 5 [18, Thm. 2.5, pp. 429, 430]. *The sets* $C \cap V_{\Phi}$, $C \cap V[v]$, *and* $C \cap \Lambda BV$ *form Banach spaces with norms*

$$\|f\|_{\Phi} = \inf\{r > 0 : v_{\Phi}(f/r) \leqslant 1\} + |f(1)|, \tag{21}$$

$$\|f\|_v = \sup_{n \in N} \frac{v(n, f)}{v(n)} + |f(1)|, \tag{22}$$

and

$$\|f\|_{\Lambda} = v_{\Lambda}(f) + |f(1)|, \tag{23}$$

respectively.

(Regarding (21) and (23) see, also, [11, p. 32] and [24, p. 108]).

DEFINITION 6 (cf. [28, p. 16]). We say that a function Φ has the complementary function Ψ in the sense of W. H. Young, if

$$\Phi(x)=\int_0^x \phi(t)\,dt \qquad \text{and} \qquad \Psi(x)=\int_0^x \psi(t)\,dt,$$

where ϕ is a strictly increasing continuous function on $[0, \infty)$, $\phi(0)=0$, and $\psi(x)=\phi^{-1}(x)$ for $x\in[0, \infty)$.

PROOFS

Proof of Theorem 1. Sufficiency. In view of Lemma 3 it is enough to show that $C\cap \Lambda^q BV\subset U$, where

$$\Lambda^q=(k^{1/2-q})_{k=1}^{\infty}, \tag{24}$$

and q is defined by (5).

Suppose $f\in C\cap \Lambda^q BV$, $x\in[-1, 1]$ is arbitrary, and $m=m_n$ is an integer defined in Lemma 2. Now set

$$f(x)-L_n(x, f)=\sum_{k=1}^{m}(f(x_{k,n})-f(x))\,l_{k,n}(x)$$
$$+\sum_{k=m+1}^{n}(f(x_{k,n})-f(x))\,l_{k,n}(x)\equiv I_1+I_2.$$

By Abel's transformation we have

$$I_1=(f(x_{m,n})-f(x))\sum_{i=1}^{m} l_{i,n}(x)+\sum_{k=1}^{m-1}(f(x_{k,n})-f(x_{k+1,n}))\sum_{i=1}^{k} l_{i,n}(x).$$

Consequently, applying (1), (16)–(18), and (20), we obtain

$$|I_1|<K\left(\omega(1/n, f)+\sum_{k=1}^{m-1}\frac{|f(x_{k,n})-f(x_{k+1,n})|}{(m-k)^{1/2-q}}\right).$$

Let us arrange the numbers $f(I_{k,n})=|f(x_{k,n})-f(x_{k+1,n})|$ in decreasing order: $f(I_{n_k,n})\geqslant f(I_{n_{k+1},n})$, $k=1, ..., n-1$. Then we have

$$|I_1|<K\left(\omega(1/n, f)+\sum_{k=1}^{m-1}\frac{f(I_{n_k,n})}{k^{1/2-q}}\right).$$

In view of the similarity between the estimations of I_1 and I_2 we omit the details of the estimation of I_2. Finally, we obtain

$$|f(x)-L_n(x,f)|<K\left(\omega(1/n,f)+\sum_{k=1}^{n-1}\frac{f(I_{n_k,n})}{k^{1/2-q}}\right).$$

Without loss of generality we can assume that $f \neq \text{const}$. Now set $e=[\omega(1/n,f)^{-1}]$, where $[a]$ means the integer part of a number a. According to (15) we have $x_{k,n}-x_{k+1,n}=O(1/n)$ $(k=1,...,n-1)$, and so, for $n>N_0$ (here and elsewhere N_0 denotes a sufficiently large positive integer), by (1) we get

$$|f(x)-L_n(x,f)|<K\left(\omega(1/n,f)\sum_{k=1}^{e}\frac{1}{k^{1/2-q}}+\sum_{k=e+1}^{n-1}\frac{f(I_{n_k,n})}{k^{1/2-q}}\right)\equiv J_1+J_2. \tag{25}$$

It is clear that since $q<1/2$, we have $J_1=o(1)$. Taking into account that the intervals $(x_{k,n}, x_{k+1,n})$, $k=1,...,n-1$, are non-overlapping, we can estimate J_2 as follows:

$$\begin{aligned} J_2 &= \sum_{k=e+1}^{n-1}\frac{f(I_{n_k,n})}{k^{1/2-q}}=\sum_{k=1}^{n-e-1}\frac{f(I_{n_{k+e},n})}{(k+e)^{1/2-q}} \\ &< \sum_{k=1}^{n-e-1}\frac{f(I_{n_k,n})}{(k+e)^{1/2-q}}\leqslant v_{\Lambda^q_e}(f). \end{aligned} \tag{26}$$

Obviously the sequence Λ^q satisfies the conditions of Lemma 4, so $f\in\Lambda^q BV=\Lambda^q_C BV$, and by Definition 5, $J_2<v_{\Lambda^q_e}(f)=o(1)$, independent of $x\in[-1,1]$. Thus, combining (25) and (26), we have $f\in U$.

Necessity. We always assume that $\alpha\geqslant\beta$ since the case $\alpha<\beta$ reduces to the previous one via identity (12). Now let $q>-1/2$, where q is defined by (5). If condition (6) is not fulfilled, then by virtue of Lemma 3 (see [16, Proof of Thm. 3, p. 116] there exists a decreasing sequence of positive numbers $(a_k)_{k=1}^{\infty}$, $a_k\to 0$, as $k\to\infty$, such that

$$\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}<\infty \tag{27}$$

and

$$\sum_{k=1}^{\infty}\frac{a_k}{k^{1/2-q}}=\infty. \tag{28}$$

Let us consider a sequence of linear functionals $L_n(f)=L_n(1, f)$ $(n\in N)$ defined on a Banach space $C\cap \Lambda BV$ with norm (23). We shall show that a sequence of norms of the functionals $(\|L_n\|)_{n=1}^{\infty}$ is not bounded. Then an existence of $f_0\in C\cap \Lambda BV$ such that $f_0\notin U$ immediately follows from Banach-Steinhaus Theorem. For this purpose let us define functions

$$f_n(x)=\begin{cases}(-1)^{k+1}a_k & \text{for} \quad x=x(k,n),\ k=1, ..., [n/2],\\ 0 & \text{for} \quad x=-1,\ 1, \text{ and } x([n/2]+1, n),\\ \text{linear} & \text{for the rest} \quad x\in[-1,1],\end{cases} \tag{29}$$

where $n>N_0$.

It follows from (27) and (29) that $f_n\in C$ and $\|f_n\|_\Lambda<K$ $(n>N_0)$. Meanwhile, combining (13), (14), (19), and taking into account that the $\operatorname{sign}(P_n'(x_{k,n}))=(-1)^{k+1}$ $(k=1, ..., n)$ [12, p. 71], we get

$$\begin{aligned} L_n(f_n)&=\sum_{k=1}^{n}\frac{f_n(x_{k,n})\,P_n(1)}{P_n'(x_{k,n})(1-x_{k,n})}\\ &>K\sum_{k=2}^{[n/2]}\frac{(-1)^{k+1}a_k n^q}{(-1)^{k+1}k^{-q-3/2}n^{q+2}(k-1)^2n^{-2}}>K\sum_{k=2}^{[n/2]}\frac{a_k}{k^{1/2-q}}. \end{aligned} \tag{30}$$

Then by (28) and (30) $\|L_n\|\geqslant|L_n(f_n)|/\|f_n\|_\Lambda\to\infty$ as $n\to\infty$, and the assertion of Theorem 1 for $q>-1/2$ is proved.

To complete the proof for the case $-1<q\leqslant-1/2$ it is sufficient to mention that by the asymptotic formula [19, Thm. 8.21.13, p. 197], $P_n^{(\alpha,\beta)}(0)>Kn^{-1/2}$ on an infinite subsequence of positive integers n. Then consider $L_n(0, g_n)$, where

$$g_n(x)=\begin{cases}(-1)^{k+1}a_{[n/2]+1-k} & \text{for} \quad x=x(k,n),\ k=[n/4], ..., [n/2],\\ 0 & \text{for} \quad x=-1,\ 1,\ x([n/4]-1, n),\\ & \text{and} \quad x([n/2]+1, n),\\ \text{linear} & \text{for the rest} \quad x\in[-1,1],\end{cases} \tag{31}$$

where $n>N_0$. ∎

Proof of Theorem 2. Sufficiency. If $v(n)$ satisfies (7), then $V[v]\subset\Lambda^q BV$ [3, Thm. 2, p. 232] (just set $\lambda_k=k^{1/2-q}$, $k\in N$). We combine this with Theorem 1 to obtain sufficiency of condition (7).

Necessity. Let us assume that condition (7) does not hold, i.e.,

$$\sum_{k=1}^{\infty}\frac{v(k)}{k^{3/2-q}}=\infty. \tag{32}$$

Applying Abel's transformation it is trivial to check that, when $q<1/2$, (32) implies

$$\sum_{k=1}^{\infty} \frac{v(k)-v(k-1)}{k^{1/2-q}}=\infty. \tag{33}$$

Again we apply an idea of unboundedness of a sequence of norms of the linear functionals $(L_n)_{n=1}^{\infty}$ defined on a Banach space $C \cap V[v]$ with norm (22). Following a construction of the counterexample of Theorem 1, for $q>-1/2$ let us consider a sequence of functions (29), where $a_k=v(k)-v(k-1)$, $k \in N$.

Since $f_n \in C$ and $\|f_n\|_v<K$ for $n \in N$, the rest of the proof follows from (30) and (33). Analogously, for the case $q=-1/2$ we consider the sequence $L_n(0, g_n)$, where the functions g_n are defined by (31). ▮

Proof of Theorem 3. Sufficiency. It is known that conditions (8) and (9) are equivalent (see [14, p. 620] and [1, Lemma 4, p. 271] for the cases $q=-1/2$ and $q>-1/2$, respectively). At the same time, condition (8) implies the following inclusion [24, Thm. 1, p. 112]: $V_\Phi \subset \Lambda^q BV$, where Λ^q is defined by (24). The rest of the proof immediately follows from Theorem 1.

Necessity. Let us assume that condition (8) does not hold, i.e.,

$$\sum_{k=1}^{\infty} \Psi(k^{q-1/2})=\infty. \tag{34}$$

As it is obvious, we consider the same sequence of linear functionals L_n, but now on a Banach space $C \cap V_\Phi$ with norm (21). Again, assuming that $q>-1/2$, where q is defined by (5), we consider the sequence of functions defined by (29), where now $a_k=\psi(k^{q-1/2})$, $k \in N$.

Since the functions f_n are continuous by construction, let us estimate $\|f_n\|_\Phi$. Obviously, by the convexity of Φ, we have

$$\begin{aligned} v_\Phi(f_n) &= \sum_{k=1}^{[n/2]-1} \Phi(\psi(k^{q-1/2})-\psi((k+1)^{q-1/2})) \\ &< \sum_{k=1}^{[n/2]-1} \Phi(\psi(k^{q-1/2}))-\Phi(\psi(k+1)^{q-1/2})<\Phi(\psi(1)), \end{aligned}$$

and consequently $\|f_n\|_\Phi<K$ for $n>N_0$. To complete the proof it suffices to estimate $L_n(f_n)$. From (30) we have

$$L_n(f_n)>K\sum_{k=1}^{[n/2]} \frac{\psi(k^{q-1/2})}{k^{1/2-q}}. \tag{35}$$

But is known that condition (8) is also equivalent to the conditions

$$\sum_{k=1}^{\infty} \frac{\psi(k^{q-1/2})}{k^{1/2-q}} < \infty \tag{36}$$

and

$$\int_0^1 \int_0^{1/\phi(x)-1} (t+1)^{4q/(1-2q)}\, dt\, dx < \infty. \tag{37}$$

For $q = -1/2$ see [14, conditions (1), (2), (4), and (5), pp. 619–620]. Regarding the case $q > -1/2$, see [1, Lemma 4, conditions (26), (27), and (32), pp. 271 and 274], setting $l = 1$, $\alpha_1 = -q - 1/2$, and $\delta = q + 1/2$.

So, (34) implies a divergence of series (36), and hence by (35), unboundedness of $L_n(f_n)$. The rest of the proof follows from Banach–Steinhaus Theorem. In case $q = -1/2$ likewise can be considered the functions g_n defined by (31) and the functionals $L_n(0, \cdot)$. ▮

Proof of Theorem 4. Again we refer to known results. As it is shown [2, Coro. 1 and Coro. 2, p. 55], if $f \in C$, then condition (7) implies (10), and the rest is an obvious consequence of Theorem 2. ▮

Proof of Theorem 5. Let us assume that $q > -1/2$, where q is defined by (5). We shall follow the construction suggested in [14, Proof of Thm. 3, p. 623]. Let $\phi(y) \equiv N(y)^{q-1/2}$ for $y > 0$, and $\phi(0) = 0$. We introduce the following notations:

$$\Phi(x) = \int_0^x \phi(t)\, dt, \qquad \psi(x) = \phi^{-1}(x), \qquad \text{and} \qquad \Psi(x) = \int_0^x \psi(t)\, dt. \tag{38}$$

Then, by virtue of equivalence of (36) and (37), from (11) we obtain

$$\sum_{k=1}^{\infty} \frac{\psi(k^{q-1/2})}{k^{1/2-q}} = \infty,$$

from which follows the existence of a sequence $(m_i)_{i=1}^{\infty}$ of strictly increasing positive integers such that

$$\sum_{k=m_i+1}^{m_{i+1}} \frac{\psi(k^{q-1/2})}{k^{1/2-q}} \to \infty \qquad \text{as} \quad i \to \infty. \tag{39}$$

Now we shall construct a sequence of positive integers $(n_i)_{i=1}^{\infty}$ and a sequence of functions $(f_i(x))_{i=1}^{\infty}$ as follows: let n_1 be such that $x(2m_2, n_1) > 0$, and let

$$f_1(x) = \begin{cases} a_k & \text{for} \quad x = x(2k, n_1),\ k = m_1 + 1, ..., m_2, \\ 0 & \text{for} \quad x = -1, 1, \text{ and the rest nodes } x(k, n_1), \\ \text{linear} & \text{for the rest} \quad x \in [-1, 1], \end{cases}$$

where $a_k = \psi(k^{q-1/2})$, $k \in N$.

If the numbers $n_1, n_2, ..., n_{i-1}$ and the functions $f_1, f_2, ..., f_{i-1}$ have already been constructed, then n_i and f_i are chosen as follows:

$$x(1, n_{i-1}) < x(2m_{i+1} + 1, n_i), \tag{40}$$

$$|L_{n_i}(F_i)| < 1; \tag{41}$$

here $F_i(x) = \sum_{s=1}^{i-1} f_s(x)$.

$$f_i(x) = \begin{cases} a_k & \text{for} \quad x = x(2k, n_i),\ k = m_i + 1, ..., m_{i+1}, \\ 0 & \text{for} \quad x = -1, 1, \text{ and for the rest nodes } x(k, n_i), \\ \text{linear} & \text{for the rest} \quad x \in [-1, 1]. \end{cases} \tag{42}$$

Inequality (40) follows from (15), and (41) is possible by virtue of Theorem 1 since $F_i \in V$ for every $i \in N$.

Let $f_0(x) = \sum_{i=1}^{\infty} f_i(x)$. Since the supports of f_i $(i \in N)$ are nonoverlapping (see (40) and (42)) and $a_k \to 0$ as $k \to \infty$, it follows from (42) that $f_0 \in C$. Regarding $N(y, f_0)$, since $0 \leqslant f_0(x) \leqslant \psi(1)$, let us assume that for a given $y \in (0, \psi(1)]$, $\psi((k+1)^{q-1/2}) < y \leqslant \psi(k^{q-1/2})$ for some $k \in N$. Then by (42)

$$N(y, f_0) \leqslant 2[(m_2 - m_1) + (m_3 - m_2) + \cdots + (k - m_i)] \leqslant 2k.$$

On the other hand (see (38))

$$(k+1)^{q-1/2} < \phi(y) \leqslant k^{q-1/2},$$

i.e., $N(y) = \phi(y)^{2/(2q-1)} \geqslant k$, and hence $N(y, f_0) \leqslant 2N(y)$.

Now let us estimate $L_n(f_0)$. If $F^i(x) = \sum_{s=i+1}^{\infty} f_s(x)$, then

$$L_{n_i}(f_0) = L_{n_i}(F_i) + L_{n_i}(f_i) + L_{n_i}(F^i) \equiv \Gamma_1 + \Gamma_2 + \Gamma_3. \tag{43}$$

By (41) $|\Gamma_1| \leqslant 1$, and by (40) $\Gamma_3 \equiv 0$. Regarding Γ_2 (see (30) and (42)) we have

$$|L_{n_i}(f_i)| > K \sum_{k=m_i+1}^{m_{i+1}} \frac{\psi(k^{q-1/2})}{k^{1/2-q}},$$

which combined with (39) and (43) implies $L_{n_i}(f_0) \to \infty$ as $i \to \infty$, so $f_0 \not\subset U$. Again, for the case $q = -1/2$ we consider the sequence of functionals $L_n(0, \cdot)$ and an obvious modification of functions (31). Thus the proof is completed. ▮

ACKNOWLEDGMENT

The author thanks Professor T. Hagstrom and Professor P. Nevai for some very helpful suggestions.

REFERENCES

1. T. I. Ahobadze, Functions of bounded generalized second variation, *Math. USSR-Sb.* **37** (1980), 261–294.
2. V. O. Asatiani and Z. A. Chanturia, The modulus of variation of a function and the Banach indicatrix, *Acta Sci. Math. (Szeged)* **45** (1983), 51–66.
3. M. Avdispahić, On the classes ΛBV and $V[v]$, *Proc. Amer. Math. Soc.* **95** (1985), 230–234.
4. S. Banach, Sur les lignes rectifiables et les surfaces dont l'aire est finie, *Fund. Math.* **7** (1925), 225–236.
5. D. L. Berman, Convergence of Lagrange interpolation process for absolutely continuous functions and for functions of bounded variation, *Dokl. Akad. Nauk SSSR* **112** (1957), 9–12.
6. Z. A. Čanturija, The modulus of variation of a function and its application in the theory of Fourier series, *Soviet Math. Dokl.* **15** (1974), 67–71.
7. Ja. L. Geronimus, On the convergence of the Lagrange interpolation process with nodes at the roots of orthogonal polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* **27** (1963), 529–560.
8. A. A. Kel'zon, Interpolation of functions of bounded p-variation, *Izv. Vuzov. Matem.* **5** (1978), 131–134.
9. A. A. Kel'zon, On interpolation of continuous functions of bounded p-variation, *Soviet Math. (Iz. VUZ.)* **8** (1984), 17–24.
10. V. I. Krylov, Convergence of algebraic interpolation with respect to roots of Čebyšev's polynomial for absolutely continuous functions and functions of bounded variation, *Dokl. Akad. Nauk SSSR* **107** (1956), 362–365.
11. J. Musielak and W. Orlicz, On generalized variations (1), *Studia Math.* **18** (1959), 11–41.
12. G. I. Natanson, Two-sided estimates for the Lebesgue function of Lagrange interpolation based on the Jacobi nodes, *Izv. Vuzov Matem.* **11** (1967), 67–74.
13. G. P. Nevai, Remarks on interpolation, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 123–144.
14. K. I. Oskolkov, Generalized variation, the Banach indicatrix, and the uniform convergence of Fourier series, *Math. Notes* **12** (1972), 619–625.
15. S. S. Pilipchuk, Tests for convergence of interpolation processes, *Izv. Vuzov. Matem.* **12** (1979), 39–44.
16. S. Perlman and D. Waterman, Some remarks on functions of Λ-bounded variation, *Proc. Amer. Math. Soc.* **74** (1979), 113–118.

17. A. I. Sablin, Λ-variation and Fourier series, *Soviet Math. (Izv. VUZ.)* **10** (1987), 87–90.
18. N. P. Schembari and M. Schramm, $\Phi V[h]$ and Riemann-Stieltjes integration, *Colloq. Math.* **60/61** (1990), 421–441.
19. G. Szegő, "Orthogonal Polynomials," Mem. Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1967.
20. X. Sun, Lagrange interpolation of functions of generalized bounded variation, *Acta Math. Hungar.* **53** (1989), 75–84.
21. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Pergamon Press, New York, 1963.
22. P. Vértesi, Lagrange interpolation for continuous functions of bounded variation, *Acta Math. Acad. Sci. Hungar.* **35** (1980), 23–31.
23. P. Vértesi, One-sided convergence conditions for Lagrange interpolation based on the Jacobi roots, *Acta Sci. Math. (Szeged)* **45** (1983), 419–428.
24. D. Waterman, On convergence of Fourier series of functions of generalized bounded variation, *Studia Math.* **44** (1972), 107–117.
25. D. Waterman, On the summability of Fourier series of functions of Λ-bounded variation, *Studia Math.* **55** (1976), 87–95.
26. N. Wiener, The quadratic variation of a function and its Fourier coefficients, *J. Math. Phys.* **3** (1924), 72–94.
27. L. C. Young, Sur une généralisation de la notion de variation de puissance p-iéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier, *C. R. Acad. Sci. Paris* **204** (1937), 470–472.
28. A. Zygmund, "Trigonometric Series," 2nd rev. ed., Vol. 1, Cambridge Univ. Press, Cambridge, 1959.